Coverings of Pairs by Quintuples

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Communicated by the Managing Editors

Received October 20, 1985

Let V be a finite set of v elements. A covering of the pairs of V by k-subsets is a family F of k-subsets of V, called blocks, such that each pair in V occurs in at least one member of F. For fixed v and k, the covering problem is to determine the number of blocks in any minimum (as opposed to minimal) covering. We will denote the number of blocks in any minimum covering by $C(v, k, 2)$. In this paper, we show that $C(v, 5, 2)$ can be determined for $v \equiv 1$ and 2 modulo 4 with a lower bound on v.

1. INTRODUCTION

Let V be a finite set of elements. A covering of the pairs of V by k-subsets is a family F of k-subsets of V, called blocks, such that each pair in V occurs in at least one member of F. For fixed v and k, the covering problem is to determine the number of blocks in any minimum (as opposed to minimal) covering. We will denote the number of blocks in any minimum covering by $C(v, k, 2)$.

A well-known lower bound for $C(v, k, 2)$ is $\lceil v\lceil (v-1)/(k-1)\rceil/k\rceil$ (Schönheim [16]). Let $L(v, k, 2)$ denote $\lceil v\lceil (v-1)/(k-1)\rceil/k\rceil$. Fort and

0097-3165/87 $3.00

Hedlund [1] have shown that $C(v, 3, 2) = L(v, 3, 2)$ for all positive integers $v \geq 3$. Recently, a simple proof of this result was provided by Stanton and Rogers [14]. It is more difficult to determine $C(v, 4, 2)$. W. H. Mills [11, 12] has proved the following: $C(v, 4, 2) = L(v, 4, 2)$ for all positive integers $v \geq 4$ and $v \neq 7, 9, 10$ or 19, $C(v, 4, 2) = L(v, 4, 2) + 1$ for $v = 7, 9$, and 10 and $C(19, 4, 2) = L(19, 4, 2) + 2$. There are only partial results known for $C(v, 5, 2)$. In this paper, we show that $C(v, 5, 2)$ can be determined for $v \equiv 1$ and 2 modulo 4 with a lower bound on v. Before summarizing the previously known results on $C(v, 5, 2)$, we note that if each pair in V occurs in precisely one block of F, the covering is called exact. An exact system is also known as a $(v, k, 1)$-BIBD.

The exact systems for block size 5 are well known. If a $(v, 5, 1)$-BIBD exists, then $C(v, 5, 2) = L(v, 5, 2)$. Hanani [4] has shown that $(v, 5, 1)$-BIBDs exist if and only if $v \equiv 1$ or $5 \pmod{20}$ and $v \geq 5$. By adding a new element to an exact system on $v - 1$ elements, it is easy to show that $C(v, 5, 2) = L(v, 5, 2)$ for $v \equiv 2$ or $6 \pmod{20}$. This is a special case of a result due to Schönheim [16]. This result has been extended [1, 2, 3, 10] and can be used to determine $C(v, 5, 2)$ in certain cases.

THEOREM 1.1. *If* $v \equiv 1 \pmod 4$ *and if* $C(v, 5, 2) = L(v, 5, 2)$, *then* $C(v + 1, 5, 2) = L(v + 1, 5, 2)$.

B. Gardner ([2], [3]) has determined that $C(v, 5, 2) = L(v, 5, 2)$ for several congruence classes of v modulo 100: $v \equiv 10, 14, 17, 18, 30, 94, 97, 98 \pmod{100}$ and $v \neq 17, 30, 94, 110, 114, 130, 194, 210, 230$. In addition, Gardner proved that $C(v, 5, 2) \geq L(v, 5, 2) + 1$ for $v \equiv 13 \pmod{20}$ and that $C(v, 5, 2) = L(v, 5, 2) + 1$ for $v \equiv 13, 93 \pmod{100}$ and $v \neq 13, 93, 113, 193, 213$. Gardner's constructions make use of the existence of three mutually orthogonal Latin squares of certain orders. Now that Wang and Wilson (see Wallis [18]) have shown that such squares exist for all orders $n > 14$, Gardner's results can be extended to include these cases except for $v = 13, 17, 30$. Thus, we have the following.

LEMMA 1.2. $C(v, 5, 2) = L(v, 5, 2)$ *for all* $v \equiv 10, 14, 17, 18, 30, 94, 97, 98 \pmod{100}$, $v \neq 17, 30$, *and* $C(v, 5, 2) = L(v, 5, 2) + 1$ *for all* $v \equiv 13$ *and* $93 \pmod{100}$, $v \neq 13$.

$C(v, 5, 2)$ is also known for all $v \leq 23$ (see recent tables in [2, 9] for complete references).

The purpose of this paper is to improve these results to determine $C(v, 5, 2)$ for $v \equiv 1$ and $2 \pmod 4$ with a lower bound on v. In the next section, recursive constructions for coverings using group divisible designs are given. We apply these results to the cases $v \equiv 9, 10, 13, 14, 17$, and $18 \pmod{20}$ in Section 5. In Section 3, a construction analogous to a construction used for the coverings of pairs with quadruples by Horton, Mullin, and Stanton [7] is given. This result is used to determine $C(v, 5, 2)$ for some small values of v which are required for the recursive constructions. Since the main construction used for $v \equiv 9$, 13, and 17 (mod 20) is recursive, the lower bounds on v in these cases could be improved if the covering numbers of some smaller values of v were known to meet the Schönheim lower bound. Unfortunately, $C(v, 5, 2) \neq L(v, 5, 2)$ for $v = 9$, 13, 17, and 29. In Section 6, we give bounds for $C(v, 5, 2)$, for $v = 29$, and $v = 53$. These results are obtained by using some special resolvable designs. Finally, we summarize the results known for $C(v, 5, 2)$ and $v \equiv 1$ and 2 (mod 4) in the last section.

2. Constructions Using Group Divisible Designs

Group divisible designs and pairwise balanced designs can be used to construct coverings. We will use the following definitions and results. V will be a finite set of v elements and K will be some subset of positive integers.

A pairwise balanced design, denoted PBD(v; K), is a collection B of subsets (called blocks) of V such that every pair of distinct elements of V is contained in precisely one block of B and for each $b \in B$, $|b| \in K$.

A group divisible design (GDD) is a collection B of subsets (called blocks) of size k, $k \in K$, taken from V along with a partition of V into groups $G_1, G_2, \ldots, G_m$ such that

(1) any two elements from distinct groups are contained in precisely λ_2 blocks of B,

(2) any two distinct elements from the same group are contained in exactly λ_1 blocks of B ($\lambda_1 < \lambda_2$).

We denote such a design by GD(v; K; $G_1, G_2, \ldots, G_m$; λ_1, λ_2). Let M be a subset of positive integers such that $|G_i| \in M$ for $i = 1, 2, \ldots, m$. In this case, we denote the design by GD(v; K; M; λ_1, λ_2). If an element of K or M is starred, this means that there is exactly one block or group of this size in the design. If $K = \{k\}$ or $M = \{m\}$, we denote the design by GD(v; k; M; λ_1, λ_2) or GD(v; K; m; λ_1, λ_2), respectively.

A transversal design with k groups of cardinality n, denoted by TD(k, n), is a GDD which has k groups of size n, block size k, $\lambda_1 = 0$ and $\lambda_2 = 1$. We will use the following lemma, which is a combination of known results [5, 15, 17, 19].

Lemma 2.1. *If n is a positive integer, $n \equiv 0$ or 1 (mod 5), $n \neq 6$, 10, 20, 26, 30, then there exists a* TD(6, n).

Group divisible designs with block size k are useful in constructing coverings of pairs with k-subsets. Let G be a GD(v; k; $G_1, G_2, \ldots, G_m$; 0, 1) defined on a v-set V and let B be the collection of blocks of G. Let C_i be a covering of the pairs of G_i with blocks of size k. Then, the collection of blocks $B \cup \bigcup_{i=1}^{m} C_i$ is a covering of the pairs of V with blocks of size k. This technique was used in determining $C(v, 4, 2)$ and in the original proof of $C(v, 3, 2)$ [10]. We apply it to constructing coverings with 5-subsets. We first construct a group divisible design with block size 5 from a TD(6, m).

LEMMA 2.2. *If there exists a* TD(6, m) *and* t *is an integer such that* $0 \leqslant t \leqslant m$, *then there exists a* GD($20m+4t$; 5; $\{4m, (4t)^*\}$; 0, 1).

Proof. Delete $m-t$ elements from one group of a TD(6, m). The resulting design D is a GD($5m+t$; $\{5, 6\}$; $\{m, t^*\}$; 0, 1). Let D be defined on the set X, where $|X| = 5m+t$. We construct E, a GD($20m+4t$; 5; $\{4m, (4t)^*\}$; 0, 1) on $X \times \{1, 2, 3, 4\}$, as follows. If G is a group of D, then $G \times \{1, 2, 3, 4\}$ is a group of E. The collection of blocks for E is obtained by replacing each block C of D with the blocks of a GD($4\,|C|$; 5; 4; 0, 1) constructed on the set $C \times \{1, 2, 3, 4\}$ with groups $\{c\} \times \{1, 2, 3, 4\}$ for $c \in C$. To construct a GD(20; 5; 4; 0, 1), we delete one element from a (21, 5, 1)-BIBD and use the resulting blocks of size 4 as the groups of the GDD. Similarly, to construct a GD(24; 5; 4; 0, 1) we delete one element from a (25, 5, 1)-BIBD. ∎

COROLLARY 2.3. *Let* m *and* t *be positive integers such that* $0 \leqslant t \leqslant m$ *and there exists a* TD(6, m). *If there is a* ($4m+1$, 5, 1)-BIBD *and* $C(4t+1, 5, 2) = L(4t+1, 5, 2) + s$ *where* s *is a nonnegative integer, then* $C(20m+4t+1, 5, 2) \leqslant L(20m+4t+1, 5, 2) + s$.

Proof. Add a new element x to all groups of the GD($20m+4t$; 5; $\{4m, (4t)^*\}$; 0, 1) constructed in Lemma 2.2. The resulting design will be a PBD($20m+4t+1$; $\{5, 4m+1, (4t+1)^*\}$) with $16m^2+8tm$ blocks of size 5. Replace each block B of size $4m+1$ with the blocks of a ($4m+1$, 5, 1)-BIBD defined on the elements of B and replace the block B' of size $4t+1$ with the $L(4t+1, 5, 2)+s$ blocks of a covering of the pairs of the elements of B' with subsets of size 5. This gives a covering of the pairs of a $20m+4t+1$ set with $20m^2+8tm+m+\lceil((4t+1)\,t/5)\rceil+s$ blocks or $L(20m+4t+1, 5, 2)+s$ blocks of size 5. ∎

COROLLARY 2.4. *Let* m *and* t *be positive integers such that* $0 \leqslant t \leqslant m$ *and there exists a* TD(6, m). *If there is a* ($4m+1$, 5, 1)-BIBD *and* $C(4t+2, 5, 2) = L(4t+2, 5, 2) + s$ *where* s *is a nonnegative integer, then* $C(20m+4t+2, 5, 2) \leqslant L(20m+4t+2, 5, 2) + s$.

Proof. Add a new element x to all groups of size $4m$ of the GD$(20m+4t;\ 5;\ \{4m,\ (4t)^*\};\ 0,\ 1)$ constructed by Lemma 2.2 and replace each of these $(4m+1)$-element sets with the blocks of a $(4m+1,\ 5,\ 1)$-BIBD. To the group of size $4t$ we add both x and an additional new element y to obtain a $(4t+2)$-element set B'. Replace B' with the $C(4t+2,\ 5,\ 2)$ blocks of a covering of its pairs by quintuples. We add $5m$ new blocks to cover the remaining pairs $\{y, \alpha\}$ with $\alpha \notin B'$. This gives a covering of the pairs of a $(20m+4t+2)$-element set with $L(20m+4t+2,\ 5,\ 2)+s$ blocks of size 5. ∎

Corollaries 2.3 and 2.4 will be the main constructions used in our applications. There are two other constructions for coverings with 5-subsets using Lemma 2.2. We state these without proof.

COROLLARY 2.5. *Let m and t be positive integers such that $0 \leqslant t \leqslant m$ and there exists a* TD$(6, m)$. *If there exists a* PBD$(4m+l;\ \{5, l^*\})$ *and a* PBD$(4t+l;\ \{5, l^*\})$ *and $C(l, 5, 2) = L(l, 5, 2)+s$ where s is a nonnegative integer, then $C(20m+4t+l,\ 5,\ 2) \leqslant L(20m+4t+l, 5, 2)+s$.*

COROLLARY 2.6. *Let m and t be positive integers such that $0 \leqslant t \leqslant m$ and there exists a* TD$(6, m)$. *Then $C(20m+4t,\ 5,\ 2) \leqslant 24mt+16m(m-t)+5C(4m,\ 5,\ 2)+C(4t,\ 5,\ 2)$.*

3. A Construction Using Incomplete Orthogonal Arrays

Let V be a finite set of size n. Let K be a subset of size k of V. An incomplete orthogonal array IA$(n,\ k,\ s)$ is an $(n^2-k^2)\times s$ array written on the symbol set V such that every ordered pair of symbols in $V\times V-(K\times K)$ occurs in any ordered pair of columns in the array. We may think of an IA$(n,\ k,\ s)$ as a set of $s-2$ mutually orthogonal Latin squares of order n which are missing a subsquare of order k. We need not be able to fill in the $k\times k$ missing subsquares with Latin squares of order k.

The following construction is analogous to a construction used by Horton, Mullin, and Stanton in [7] for covering the pairs of a v element set with quadruples for $v\equiv 10$ and $11 \pmod{12}$. We require the existence of resolvable $(v,\ 4,\ 1)$-BIBDs. A necessary condition for the existence of a $(v,\ 4,\ 1)$-RBIBD is $v\equiv 4 \pmod{12}$. Hanani, Ray-Chaudhuri, and Wilson [6] have shown that this is also sufficient.

LEMMA 3.1. *Let n, a, and s be nonnegative integers and $0 \leqslant a \leqslant 4n+1$. If there exists an* IA$(12n+4+a,\ a,\ 5)$ *and if $C(4n+4a+1,\ 5,\ 2)=$*

$L(4n+4a+1,\ 5,\ 2)+s$, *then* $C(64n+4a+21,\ 5,\ 2) \leqslant L(64n+4a+21,\ 5,\ 2)+s$.

Proof. Let $V=\{1, 2, ..., 12n+4\}$ and $A=\{\bar{1}, \bar{2}, ..., \bar{a}\}$. Let $V_i=\{1_i, 2_i, ..., (12n+4)_i\}$ and $A_i=\{\bar{1}_i, \bar{2}_i, ..., \bar{a}_i\}$ for $i=1, 2, ..., 5$.

Let D be a $(12n+4, 4, 1)$-RBIBD defined on V and let $R_1, R_2, ..., R_{4n+1}$ denote the resolution classes of D. If $a>0$, add a new element $\bar{i}$ to each quadruple in R_i for $i=1, 2, ..., a$. This construction yields a system $S(V; A)$ consisting of blocks of size 4 and 5 with the property that every pair of elements from $V \cup A$ occurs in some block of the system with the exception of the pairs $\{\bar{i}, \bar{j}\}$ for $\bar{i}, \bar{j} \in A$.

We next use the same construction to build five systems $S_j(V_j;\ A_j)$ for $j=1, 2, ..., 5$. For $i=a+1, ..., 4n+1$, add a new element $\bar{i}$ to each quadruple in each of the five copies of R_i. This results in a collection B_1 of $5(3n+1)(4n+1)$ blocks of size 5 with the property that every pair of elements from $V_i \cup A_i$ occurs in some block of B_1 with the exception of the pairs $\{\alpha_i, \beta_i\}$ for $\alpha_i, \beta_i \in A_i$ and $1 \leqslant i \leqslant 5$. We must also cover all pairs of different subscripted elements.

We use the IA$(12n+4+a,\ a,\ 5)$ to cover the pairs of the form $\{\alpha_i, \alpha_j\}$ for $\alpha_i \in V_i$, $\alpha_j \in V_j$ $(i \neq j)$ and $\{\alpha_i, \beta_j\}$ for $\alpha_i \in V_i$, $\beta_j \in A_j$ $(i \neq j)$ with blocks of size 5. We will consider the IA$(12n+4+a,\ a,\ 5)$ as three mutually orthogonal Latin squares of side $12n+4+a$ which are missing a subsquare of side a in the upper left-hand corner. (The subsquare of side a need not exist.) Let M_1, M_2, and M_3 be the three mutually orthogonal Latin squares and let M_i be written on the symbol set $V_i \cup A_i$ where the subsquare of side a is defined on the symbol set A_i. Let M be the superposition of the arrays M_1, M_2, and M_3. Index the rows of M by the elements of $A_4 \cup V_4$ and the columns of M by the elements of $A_5 \cup V_5$. Add the elements of $A_4 \cup V_4$ and $A_5 \cup V_5$ to M as follows:

1. If a cell in the row labeled x is nonempty, add x to each triple in that row.

2. If a cell in the column labeled y is nonempty, add y to each subset in that column.

This construction yields a collection B_2 of $(12n+4+a)^2-a^2$ blocks of size 5 with the property that each of the pairs $\{\alpha_i, \alpha_j\}$ for $\alpha_i \in V_i$, $\alpha_j \in V_j$ $(i \neq j)$ and $\{\alpha_i, \beta_j\}$ for $\alpha_i \in V_i$, $\beta_j \in A_j$ $(i \neq j)$ occurs in some block of B_2.

To cover the remaining pairs of $\bigcup_{i=1}^{5} (V_i \cup A_i) \cup \{\overline{a+1}, ..., \overline{4n+1}\}$, we take the blocks of a covering of the pairs of the set $\bigcup_{i=1}^{5} A_i \cup \{\overline{a+1}, \overline{a+2}, ..., \overline{4n+1}\}$ with blocks of size 5.

We have now constructed a covering of the pairs of $\bigcup_{i=1}^{5} (V_i \cup A_i) \cup \{\overline{a+1}, ..., \overline{4n+1}\}$ with $|B_1|+|B_2|+C(4n+4a+1,\ 5,\ 2)$ blocks of size 5.

Let $C(4n+4a+1, 5, 2) = L(4n+4a+1, 5, 2)+s$. It is easy to verify that $C(64n+4a+21, 5, 2) \leqslant |B_1| + |B_2| + C(4n+4a+1, 5, 2) \leqslant L(64n+4a+21, 5, 2)+s$. ∎

A modification of the construction in Lemma 3.1 leads to the following.

Corollary 3.2. *Let n, a, and s be nonnegative integers and $0 \leqslant a \leqslant 4n+1$. If there exists an* $\mathrm{IA}(12n+4+a, a, 5)$ *and if* $C(4n+4a+2, 5, 2) = L(4n+4a+2, 5, 2)+s$, *then* $C(64n+4a+22, 5, 2) \leqslant L(64n+4a+22, 5, 2)+s$.

Proof. Let V_i, A_i $(i=1, 2, ..., 5)$ and B_1 and B_2 be as defined in the proof of Lemma 3.1. Let $W = \bigcup_{i=1}^{5} (V_i \cup A_i) \cup \{\overline{a+1}, ..., \overline{4n+1}\} \cup \{\infty\}$. Let B_3 be the $C(4n+4a+2, 5, 2)$ blocks of size 5 of a covering of the pairs of $\bigcup_{i=1}^{5} A_i \cup \{\overline{a+1}, ..., \overline{4n+1}\} \cup \{\infty\}$. Partition the elements of $\bigcup_{i=1}^{5} V_i$ into subsets of size 4 and add the element ∞ to each of these quadruples. This will give a collection B_4 of $15n+5$ blocks of size 5 which contain each of the pairs $\{\infty, \alpha\}$ for $\alpha \in \bigcup_{i=1}^{5} V_i$. The collection of blocks $B_1 \cup B_2 \cup B_3 \cup B_4$ is a covering of the pairs of a $64n+4a+22$ set with $L(64n+4a+22, 5, 2)+s$ blocks of size 5. Thus, $C(64n+4a+22, 5, 2) \leqslant L(64n+4a+22, 5, 2)+s$. ∎

We use Lemma 3.1 and Corollary 3.2 to determine $C(v, 5, 2)$ for some small values of v which we need in Section 5.

LEMMA 3.3. $C(v, 5, 2) = L(v, 5, 2)$ *for* $v = 90, 150, 154, 158$ *and* 557.

Proof. Since there exists an $\mathrm{IA}(17, 1, 5)$ and $C(10, 5, 2) = L(10, 5, 2)$, by Corollary 3.2 with $n=a=1$, $C(90, 5, 2) = L(90, 5, 2)$. By applying Corollary 3.2 with $n=2$, and $n=2$, and $a=1$ we show that $C(v, 5, 2) = L(v, 5, 2)$ for $v = 150$ and 154, respectively. Since there exists an $\mathrm{IA}(30, 2, 5)$ [20] and $C(18, 5, 2) = L(18, 5, 2)$, by Corollary 3.2 with $n=a=2$, $C(158, 5, 2) = L(158, 5, 2)$. Finally, let $n=7$ and $a=22$ in Lemma 3.1. There exists an $\mathrm{IA}(110, 22, 5)$ and $C(117, 5, 2) = L(117, 5, 2)$ by Lemma 1.2. Thus, we have $C(557, 5, 2) = L(557, 5, 2)$. ∎

4. DIRECT CONSTRUCTIONS

In this section we will use direct constructions to show that $C(v, 5, 2) = L(v, 5, 2)$ for ten values of v.

LEMMA 4.1. $C(v, 5, 2) = L(v, 5, 2)$ *for* $v = 30, 34, 50, 58, 74, 78, 138, 170, 174,$ *and* 178.

Proof. "$v=30$" We have $L(30, 5, 2)=48$. We take our set to be the set of all pairs (i, j) where i is an integer modulo 15 and j is 1 or 2. We take the 48 blocks:

$$\begin{array}{llllll} (0,1) & (3,1) & (6,1) & (9,1) & (12,1) & \text{mod } (15,\text{-}) \text{ period } 3 \\ (0,1) & (0,2) & (1,2) & (3,2) & (7,2) & \text{mod } (15,\text{-}) \\ (0,1) & (1,1) & (5,1) & (10,2) & (13,2) & \text{mod } (15,\text{-}) \\ (0,1) & (2,1) & (8,1) & (4,2) & (14,2) & \text{mod } (15,\text{-}). \end{array}$$

Here the notation (x, y) mod $(w,-)$ means that y is fixed and that we add an arbitrary constant modulo w to x. These 48 blocks cover all pairs so that $C(30, 5, 2)=48$.
"$v=34$" We have $L(34, 5, 2)=62$. Delete one element from a TD(5, 7) to obtain a GDD on 34 elements. Let the points of this GDD be the set of all pairs (i, j), $0 \leqslant i \leqslant 4$, $0 \leqslant j \leqslant 6$, $(i, j) \neq (4, 6)$. We take the group G_i to consist of the points (i, j). Thus G_4 has 6 points and the other groups have 7 points each. Of the 49 blocks in the GDD, 42 are quintuples and the other 7 are quadruples. We arrange the design so that the 7 quadruples are

$$(0,j) \quad (1,j) \quad (2,j) \quad (3,j),$$

with $0 \leqslant j \leqslant 6$.

We will give a set of 60 quintuples and two quadruples that covers all pairs of this GDD. The 60 quintuples are the 42 quintuples from the GDD and the following 18 quintuples

$$\begin{array}{llllll} (i,0) & (i,1) & (i,2) & (i,3) & (i,4) & 1 \leqslant i \leqslant 4 \\ (i,0) & (i,1) & (i,2) & (i,5) & (i,6) & 1 \leqslant i \leqslant 3 \\ (i,3) & (i,4) & (i,5) & (i,6) & (0,6) & 1 \leqslant i \leqslant 3 \\ (4,0) & (4,1) & (4,2) & (4,3) & (4,5) & \\ (4,4) & (4,5) & (1,6) & (2,6) & (3,6) & \\ (0,0) & (0,1) & (0,4) & (0,5) & (0,6) & \\ (0,2) & (0,3) & (0,4) & (0,5) & (0,6) & \\ (0,0) & (1,0) & (2,0) & (3,0) & (0,2) & \\ (0,1) & (1,1) & (2,1) & (3,1) & (0,3) & \\ (0,2) & (1,2) & (2,2) & (3,2) & (0,1) & \\ (0,3) & (1,3) & (2,3) & (3,3) & (0,0) & \end{array}$$

The two quadruples are

$$\begin{array}{llll} (0,4) & (1,4) & (2,4) & (3,4) \\ (0,5) & (1,5) & (2,5) & (3,5). \end{array}$$

This collection of 60 quintuples and 2 quadruples covers all pairs so that $C(34, 5, 2) = 62$.

"$v = 50$" We have $L(50, 5, 2) = 130$. Here we use the set of 50 pairs (i, j) where i is an integer modulo 25 and j is 1 or 2. We take the 130 blocks:

(0, 1)	(5, 1)	(10, 1)	(15, 1)	(20, 1)	mod (25,-) period 5
(0, 1)	(1, 1)	(3, 1)	(10, 2)	(16, 2)	mod (25,-)
(0, 1)	(4, 1)	(13, 1)	(10, 2)	(24, 2)	mod (25,-)
(0, 1)	(6, 1)	(14, 1)	(8, 2)	(18, 2)	mod (25,-)
(0, 1)	(7, 1)	(5, 2)	(17, 2)	(21, 2)	mod (25,-)
(0, 1)	(0, 2)	(1, 2)	(3, 2)	(8, 2)	mod (25,-) .

These 130 blocks cover all pairs so that $C(50, 5, 2) = 130$.

"$v = 58$" We have $L(58, 5, 2) = 174 = 3 \cdot 58$. We take our set to be the integers modulo 58 and use the 174 blocks

0	1	2	8	19	mod 58
0	3	12	32	36	mod 58
0	5	15	28	42	mod 58

These 174 blocks cover all pairs so that $C(58, 5, 2) = 174$.

"$v = 74$" We have $L(74, 5, 2) = 282$. We take the set of pairs (i, j) where i is an integer modulo 35 and j is 1 or 2. To this set we adjoin four special elements W, X, Y, Z. We will give a collection of 281 quintuples and one quadruple that covers all pairs of this 74 element set. The quintuples are

X	Z	$(0, 1)$	$(0, 2)$	$(13, 2)$	
W	$(2i, 1)$	$(2i+1, 1)$	$(2i, 2)$	$(2i+1, 2)$	$i = 0, 1, \ldots, 17$
X	$(2i+1, 1)$	$(2i+2, 1)$	$(2i+1, 2)$	$(2i+2, 2)$	$i = 0, 1, \ldots, 16$
Y	$(4i, 1)$	$(4i+2, 1)$	$(4i+13, 2)$	$(4i+15, 2)$	$i = 0, 1, \ldots, 17$
Z	$(4i+2, 1)$	$(4i+4, 1)$	$(4i+15, 2)$	$(4i+17, 2)$	$i = 0, 1, \ldots, 16$
(0, 1)	(3, 1)	(7, 1)	(12, 1)	(22, 1)	mod (35,–)
(0, 1)	(6, 1)	(2, 2)	(10, 2)	(23, 2)	mod (35,–)
(0, 1)	(8, 1)	(16, 2)	(28, 2)	(33, 2)	mod (35,–)
(0, 1)	(11, 1)	(14, 2)	(24, 2)	(30, 2)	mod (35,–)
(0, 1)	(14, 1)	(6, 2)	(21, 2)	(32, 2)	mod (35,–)
(0, 1)	(17, 1)	(22, 2)	(26, 2)	(29, 2)	mod (35,–)

and the quadruple is $W\,X\,Y\,Z$. This collection covers all pairs so that $C(74, 5, 2) = 282$.

"$v = 78$" We have $L(78, 5, 2) = 312 = 4 \cdot 78$. We use the set of integers modulo 78 and the 312 blocks

0	2	5	9	20	(mod 78)
0	1	14	42	48	(mod 78)
0	2	21	45	53	(mod 78)
0	10	22	39	62	(mod 78)

These 312 blocks cover all pairs so that $C(78, 5, 2) = 312$.

"$v = 138$" We have $L(138, 5, 2) = 966$. We use the set of integers modulo 138 and the 966 blocks

0	1	49	76	130	(mod 138)
0	2	23	24	69	(mod 138)
0	35	94	109	122	(mod 138)
0	5	47	77	102	(mod 138)
0	70	73	107	126	(mod 138)
0	7	95	128	134	(mod 138)
0	60	86	100	118	(mod 138)

These 966 blocks cover all pairs, so that $C(138, 5, 2) = 966$.

"$v = 170$" We have $L(170, 5, 2) = 1462$. We use the set of integers modulo 170 and the 1462 blocks

$2i$	$2i+34$	$2i+68$	$2i+102$	$2i+136$	$i = 0, 1, \ldots, 16$
$2i+1$	$2i+12$	$2i+35$	$2i+58$	$2i+69$	$i = 0, 1, \ldots, 84$
0	1	7	10	88	(mod 170)
0	69	71	109	129	(mod 170)
0	86	107	123	156	(mod 170)
0	18	22	54	126	(mod 170)
0	35	43	85	140	(mod 170)
0	52	119	145	158	(mod 170)
0	17	45	76	91	(mod 170)
0	19	24	114	141	(mod 170)

These 1462 blocks cover all pairs so that $C(170, 5, 2) = 1462$.

"$v = 174$" We have $L(174, 5, 2) = 1532$. We take the set of integers modulo 170. To this set we adjoin four special elements W, X, Y, Z. We will give a collection of 1530 quintuples and two quadruples that covers all pairs of this 174 element set. The quintuples are

W	$2i$	$2i+1$	$2i+85$	$2i+86$	$i=0, 1,\ldots, 42$
X	$2i+1$	$2i+2$	$2i+86$	$2i+87$	$i=0, 1,\ldots, 41$
Y	$6i$	$6i+3$	$6i+85$	$6i+88$	$i=0, 1,\ldots, 42$
Z	$6i+3$	$6i+6$	$6i+88$	$6i+91$	$i=0, 1,\ldots, 41$
0	37	44	92	102	(mod 170)
0	12	103	109	156	(mod 170)
0	5	18	111	139	(mod 170)
0	35	39	143	159	(mod 170)
0	19	21	90	119	(mod 170)
0	17	41	130	155	(mod 170)
0	34	54	128	161	(mod 170)
0	22	52	60	147	(mod 170)

and the quadruples are $W\,X\,Y\,Z$ and $X\,Z\,0\,85$. This collection covers all pairs so that $C(174, 5, 2)=1532$.

"$v=178$" We have $L(178, 5, 2)=1602$. We use the set of integers modulo 178 and the 1602 blocks

0	1	61	91	98	(mod 178)
0	2	68	96	154	(mod 178)
0	3	8	137	166	(mod 178)
0	4	39	89	172	(mod 178)
0	9	32	109	131	(mod 178)
0	11	75	113	138	(mod 178)
0	13	27	46	157	(mod 178)
0	15	31	74	136	(mod 178)
0	17	72	124	142	(mod 178)

These 1602 blocks cover all pairs so that $C(178, 5, 2)=1602$. ∎

5. Applications

In this section, we apply the results of the previous section to determine $C(v, 5, 2)$ for $v\equiv 9$, 10, 13, 14, 17, and 18 (mod 20) with a lower bound on v in each case. The main constructions used, Corollaries 2.3 and 2.4, are recursive and we will require the covering number $C(v, 5, 2)$ for some small values of v. The following result will be useful.

LEMMA 5.1. *Let s and n be nonnegative integers:*

(i) *If* $C(4n+1, 5, 2) = L(4n+1, 5, 2) + s$, *then* $C(16n+5, 5, 2) \leqslant L(16n+5, 5, 2) + s$.

(ii) *If* $C(4n+2, 5, 2) = L(4n+2, 5, 2) + s$, *then* $C(16n+6, 5, 2) \leqslant L(16n+5, 5, 2) + s$.

(iii) *If* $C(20n+18, 5, 2) = L(20n+18, 5, 2)$, *then* $C(100n+90, 5, 2) = L(100n+90, 5, 2)$.

Proof. Let n be a positive integer and $V = \{1, 2, ..., 12n+4\}$. Let D be a $(12n+4, 4, 1)$-RBIBD defined on V and let $R_1, R_2, ..., R_{4n+1}$ denote the resolution classes of D. (See [6] for the existence of $(12n+4, 4, 1)$-RBIBDs.) Add a new element α_i to each quadruple in the resolution class R_i for $i = 1, 2, ..., 4n+1$. Let B_1 denote the resulting collection of 5-subsets. Let B_2 be the collection of blocks of a covering of the pairs of $\{\alpha_1, \alpha_2, ..., \alpha_{4n+1}\}$ with $L(4n+1, 5, 2) + s$ blocks of size 5. Then $B_1 \cup B_2$ is a covering of the pairs of $V \cup \{\alpha_1, \alpha_2, ..., \alpha_{4n+1}\}$ with $L(16n+6, 5, 2) + s$ blocks of size 5.

Let X be an additional element, and let B_3 be the collection of blocks of a covering of the pairs of $\{\alpha_1, \alpha_2, ..., \alpha_{4n+1}, X\}$ with $L(4n+2, 5, 2) + s$ blocks of size 5. Let B_4 be the set of $3n+1$ blocks of size 5 obtained by adjoining X to each quadruple in one of the resolution classes, say R_1. Then $B_1 \cup B_3 \cup B_4$ is a covering of the pairs of $V \cup \{\alpha_1, \alpha_2, ..., \alpha_{4n+1}, X\}$ with $L(16n+6) + s$ blocks of size 5.

Gardner [2] has shown that if $C(20n+18, 5, 2) = L(20n+18, 5, 2)$ and if there exist three mutually orthogonal Latin squares of order $20n+18$, then $C(100n+90, 5, 2) = L(100n+90, 5, 2)$. Since Wang and Wilson (see [18]) have shown that such mutually orthogonal Latin squares exist, (iii) follows immediately. ∎

LEMMA 5.2. $C(v, 5, 2) = L(v, 5, 2)$ *for* $v = 38, 54, 70, 134, 278, 389, 390, 469$, *and* 2229.

Proof. We apply Lemma 5.1 part (ii) with $s = 0$ and $n = 2, 3, 4, 8, 24$, and part (i) with $s = 0$ and $n = 24, 29, 139$. Here Lemmas 1.2, 3.3, and 4.1 give us $C(4n+2, 5, 2) = L(4n+2, 5, 2)$ for $n = 2, 3, 4, 8, 24$ and $C(4n+1, 5, 2) = L(4n+1, 5, 2)$ for $n = 24, 29, 139$. This gives us our result for $v \neq 278$. We now have $C(70, 5, 2) = L(70, 5, 2)$ so that we can apply Lemm 5.1 (ii) again with $n = 17$ to obtain $C(278, 5, 2) = L(278, 5, 2)$. ∎

LEMMA 5.3. $C(v, 5, 2) = L(v, 5, 2) + 1$ *for* $v = 373$ *and* 453.

Proof. Using Lemma 1.2, we apply Lemma 5.1 (i) with $s = 1$ and $n = 23$ and 28 to obtain the result. ∎

We first consider the case $v \equiv 9 \pmod{20}$. Recall that a TD$(6, m)$ exists for $m \equiv 0$ or $1 \pmod 5$, $m \geqslant 31$ (Lemma 2.1).

LEMMA 5.4. *Let n be a positive integer.*

(i) *If* $n \geqslant 24$, *then* $C(100n+9, 5, 2) = L(100n+9, 5, 2)$.

(ii) *If* $n \geqslant 134$, *then* $C(100n+29, 5, 2) = L(100n+29, 5, 2)$.

(iii) *If* $n \geqslant 134$, *then* $C(100n+49, 5, 2) = L(100n+49, 5, 2)$.

(iv) *If* $n \geqslant 28$, *then* $C(100n+69, 5, 2) = L(100n+69, 5, 2)$.

(v) *If* $n \geqslant 23$, *then* $C(100n+89, 5, 2) = L(100n+89, 5, 2)$.

Proof. To prove (i), let $t = 97$ and $m = 5y-4$ where $y \geqslant 21$ in Corollary 2.3. Since there is a $(20(y-1)+5, 5, 1)$-BIBD for $y \geqslant 21$ and $C(389, 5, 2) = L(389, 5, 2)$ (Lemma 5.2), $C(100y+309, 5, 2) = L(100y+309, 5, 2)$. Similarly, we apply Corollary 2.3 with the following parameters to prove (ii)–(v): (ii) $t = 557$ and $m = 5y$ where $y \geqslant 112$, (iii) $t = 557$ and $m = 5y+1$ where $y \geqslant 112$, (iv) $t = 117$ and $m = 5y$ where $y \geqslant 24$ and (v) $t = 97$ and $m = 5y$ where $y \geqslant 20$. ▮

Combining these results, we have the following.

THEOREM 5.5. *If* $v \equiv 9 \pmod{20}$ *and* $v \geqslant 13,400$, *then* $C(v, 5, 2) = L(v, 5, 2)$.

THEOREM 5.6. *If* $v \equiv 10 \pmod{20}$, $v \neq 270$, *then* $C(v, 5, 2) = L(v, 5, 2)$.

Proof. From Lemmas 1.2, 3.3, 4.1, and 5.2 we get $C(v, 5, 2) = L(v, 5, 2)$ for all $v \equiv 10$ or $30 \pmod{100}$ as well as for $v = 50$, 70, 90, 150, 170, and 390. By Lemma 5.1 (ii) with $n = 2$ and Lemma 4.1 we get $C(38, 5, 2) = L(38, 5, 2)$ and $C(58, 5, 2) = L(58, 5, 2)$. Lemma 5.1 (iii), with $n = 1$ and 2, now gives us $C(190, 5, 2) = L(190, 5, 2)$ and $C(290, 5, 2) = L(290, 5, 2)$. Using Lemma 2.1, we apply Corollary 2.4 with (i) $t = 7$ and $m = 5y+1$, $y \geqslant 2$, $y \neq 5$, (ii) $t = 12$ and $m = 5y+1$, $y \geqslant 3$, $y \neq 5$, (iii) $t = 17$ and $m = 5y+1$, $y \geqslant 4$, $y \neq 5$. Finally we apply the same corollary with $t = 12$, 17, 22 and $m = 25$ to obtain $C(v, 5, 2) = L(v, 5, 2)$ for $v = 550$, 570, 590. ▮

LEMMA 5.7. *Let n be a positive integer.*

(i) *If* $n \geqslant 7$, *then* $C(100n+33, 5, 2) = L(100n+33, 5, 2)+1$.

(ii) *If* $n \geqslant 27$, *then* $C(100n+53, 5, 2) = L(100n+53, 5, 2)+1$.

(iii) *If* $n \geqslant 22$, then $C(100n+73, 5, 2) = L(100n+73, 5, 2)+1$.

Proof. Apply Corollary 2.3 with (i) $t = 28$ and $m = 5y+1$ where $y \geqslant 6$, (ii) $t = 113$ and $m = 5y$ where $y \geqslant 23$ and (iii) $t = 93$ and $m = 5y$ where $y \geqslant 19$. ▮

Combining Lemma 5.7, Lemma 1.2, and Gardner's results for $v \equiv 13 \pmod{20}$, we have the following:

THEOREM 5.8. *If* $v \equiv 13 \pmod{20}$ *and* $v \geqslant 2,700$, *then* $C(v, 5, 2) = L(v, 5, 2) + 1$.

The next case to consider is $v \equiv 14 \pmod{20}$. Using Lemma 1.2, we have $C(v, 5, 2) = L(v, 5, 2)$ for $v \equiv 14$ and $94 \pmod{100}$.

LEMMA 5.9. *Let n be a nonnegative integer.*

(i) *If* $n \geqslant 0$, *then* $C(100n + 34, 5, 2) = L(100n + 34, 5, 2)$.

(ii) *If* $n \geqslant 0$, *then* $C(100n + 54, 5, 2) = L(100n + 54, 5, 2)$.

(iii) *If* $n \geqslant 0$, $n \neq 2$, *then* $C(100n + 74, 5, 2) = L(100n + 74, 5, 2)$.

Proof. By Lemmas 1.2, 3.3, 4.1, and 5.2 we have $C(v, 5, 2) = L(v, 5, 2)$ for $v = 14, 34, 54, 74, 134, 154, 174$. We now apply Corollary 2.4 with (i) $t = 3$ and $m = 5y + 1$ where $y \geqslant 2$ and $y \neq 5$, (ii) $t = 8$ and $m = 5y + 1$ where $y \geqslant 2$ and $y \neq 5$, and (iii) $t = 13$ and $m = 5y + 1$ where $y \geqslant 3$ and $y \neq 5$. Applying Corollary 2.4 with $m = 25$, $t = 8$, 13 and 18, we determine $C(534, 5, 2) = L(534, 5, 2)$, $C(554, 5, 2) = L(554, 5, 2)$, and $C(574, 5, 2) = L(574, 5, 2)$. ∎

Combining the above results, we have the following:

THEOREM 5.10. *If* $v \equiv 14 \pmod{20}$, $v \neq 274$, *then* $C(v, 5, 2) = L(v, 5, 2)$.

The last case to consider for $v \equiv 1 \pmod 4$ is $v \equiv 17 \pmod{20}$. The next lemma takes care of the cases not done by Gardner.

LEMMA 5.11. *Let n be a positive integer.*

(i) *If* $n \geqslant 7$, *then* $C(100n + 37, 5, 2) = L(100n + 37, 5, 2)$.

(ii) *If* $n \geqslant 33$, *then* $C(100n + 57, 5, 2) = L(100n + 57, 5, 2)$.

(iii) *If* $n \geqslant 33$, *then* $C(100n + 77, 5, 2) = L(100n + 77, 5, 2)$.

Proof. Apply Corollary 2.3 with (i) $t = 29$ and $m = 5y + 1$ where $y \geqslant 6$, (ii) $t = 139$ and $m = 5y$ where $y \geqslant 28$ and (iii) $t = 139$ and $m = 5y + 1$ where $y \geqslant 28$. ∎

Combining this result with Lemma 1.2, we have the following:

THEOREM 5.12. *If* $v \equiv 17 \pmod{20}$ *and* $v \geqslant 3300$, *then* $C(v, 5, 2) = L(v, 5, 2)$.

Finally, we consider the case $v \equiv 18 \pmod{20}$.

THEOREM 5.13. *If* $v \equiv 18 \pmod{20}$, *then* $C(v, 5, 2) = L(v, 5, 2)$.

Proof. From Lemmas 1.2, 3.3, 4.1, and 5.2 we get $C(v, 5, 2) = L(v, 5, 2)$ for all $v \equiv 18$ or $98 \pmod{100}$ as well as for $v = 38$, 58, 78, 138, 158, 178, and 278. Using Lemma 2.1, we apply Corollary 2.4 with (i) $t = 4$ and $m = 5y + 1$, $y \geqslant 2$, $y \neq 5$, (ii) $t = 9$ and $m = 5y + 1$, $y \geqslant 2$, $y \neq 5$, (iii) $t = 14$ and $m = 5y + 1$, $y \geqslant 3$, $y \neq 5$. Finally we apply the same corollary with $t = 9$, 14, 19 and $m = 25$ to obtain $C(v, 5, 2) = L(v, 5, 2)$ for $v = 538$, 558, 578. ∎

It should be noted that since the main construction used in this section is recursive, the lower bounds on v could be improved if the covering numbers of some smaller values of v were known to meet the Schönheim lower bound. Unfortunately, for the smallest values of v in each of the cases $v \equiv 9$, 13 and $17 \pmod{20}$, the Schönheim bound is not met. We have $C(9, 5, 2) = L(9, 5, 2) + 1$ and $C(v, 5, 2) = L(v, 5, 2) + 2$ for $v = 13$ and 17. In Section 6, we discuss the best known bounds for $v = 29$ and $v = 53$. To illustrate how the bounds could be improved, suppose $C(37, 5, 2) = L(37, 5, 2)$. Applying Lemma 5.1 and Corollary 2.3 with $t = 37$ and $m = 5y$ where $y \geqslant 8$ yields $C(100n + 49, 5, 2) = L(100n + 49, 5, 2)$ for $n \geqslant 9$.

6. BOUNDS FOR $C(v, 5, 2)$ USING BIBDS WITH MULTIPLE RESOLUTIONS

In this section, we determine bounds for $C(29, 5, 2)$ and $C(53, 5, 2)$. These results are obtained by using balanced incomplete block designs with multiple resolutions. (For definitions and results on resolvable designs, see [6] or [8].) We include a generalization of the construction used to bound $C(53, 5, 2)$.

$\infty 0\bar{0}$ 412			$65\bar{1}$		$3\overline{46}$	$\overline{23}5$
$\overline{34}6$	$\infty 1\bar{1}$ 523			$06\bar{2}$		$4\overline{50}$
$5\bar{6}\bar{1}$	$\overline{45}0$	$\infty 2\bar{2}$ 634			$10\bar{3}$	
	$6\overline{02}$	$\bar{5}\bar{6}1$	$\infty 3\bar{3}$ 045			$21\bar{4}$
$32\bar{5}$		$0\overline{13}$	$\overline{60}2$	$\infty 4\bar{4}$ 156		
	$43\bar{6}$		$1\overline{24}$	$\overline{01}3$	$\infty 5\bar{5}$ 260	
		$54\bar{0}$		$2\overline{35}$	$\overline{12}4$	$\infty 6\bar{6}$ 301

FIG. 6.1. A resolvable (15, 3, 1)-BIBD.

LEMMA 6.1. $L(29, 5, 2) < C(29, 5, 2) \leqslant L(29, 5, 2) + 3$.

Proof. Gardner ([2, 3]) has shown that $C(29, 5, 2) > L(29, 5, 2)$. Let A be the 7×7 array of triples on the symbol set $V = \{0, 1, ..., 6, \bar{0}, \bar{1}, ..., \bar{6}\} \cup \{\infty\}$ in Fig. 6.1. The collection of triples obtained from the nonempty cells of A is a resolvable (15, 3, 1)-BIBD. The rows of A form one resolution of the design and the columns of A form a second resolution.

Add a new element x_i to each triple in column i of A for $i = 1, 2, ..., 7$. Add a new element y_i to each subset in row i of A for $i = 1, 2, ..., 7$. Let $W = V \cup \{x_1, x_2, ..., x_7\} \cup \{y_1, y_2, ..., y_7\}$. This construction yields 35 blocks of size 5 on W. Denote this collection of blocks by B_1. Let B_2 be the following collection of 9 blocks of size 5.

$$\{y_6, y_1, x_2, x_3, x_5\}, \quad \{y_7, y_2, x_3, x_4, x_6\}, \quad \{y_1, y_3, x_4, x_5, x_7\},$$
$$\{y_1, y_4, x_1, x_5, x_6\}, \quad \{y_1, y_5, x_2, x_6, x_7\}, \quad \{y_2, y_6, x_1, x_3, x_7\},$$
$$\{y_1, y_7, x_1, x_2, x_4\}, \quad \{y_1, y_2, y_3, y_4, y_5\}, \quad \{y_6, y_7, y_4, y_5, y_3\}.$$

It is easy to verify that $B_1 \cup B_2$ is a covering of the pairs of W with 44 blocks of size 5. Since $L(29, 5, 2) = 41$, we have $L(29, 5, 2) < C(29, 5, 2) \leqslant L(29, 5, 2) + 3$. ∎

LEMMA 6.2. $L(53, 5, 2) + 1 \leqslant C(53, 5, 2) \leqslant L(53, 5, 2) + 2$.

Proof. Let S be the array in Fig. 6.2. S is a doubly resolvable (27, 3, 1)-BIBD defined on $V = \{1, 2, ..., 27\}$.

Add a new element x_i to each subset in row i of S for $i = 1, 2, ..., 13$ and add a new element y_i to each subset in column i of S for $i = 1, 2, ..., 13$. Let $W = V \cup \{x_1, x_2, ..., x_{13}\} \cup \{y_1, y_2, ..., y_{13}\}$. Then $|W| = 53$. This construction yields 117 blocks of size 5 on W. Denote this collection of blocks by B_1.

Let B_2 be the following collection of 13 blocks on the symbols $\{x_1, x_2, ..., x_{13}\} \cup \{y_1, y_2, ..., y_{13}\}$.

$$\{x_1, y_5, y_6, y_{10}, y_{13}\}, \quad \{x_2, y_3, y_9, y_{10}, y_{12}\}, \quad \{x_3, y_2, y_8, y_{10}, y_{11}\},$$
$$\{x_4, y_2, y_3, y_7, y_{13}\}, \quad \{x_5, y_6, y_7, y_9, y_{11}\}, \quad \{x_6, y_5, y_7, y_8, y_{12}\},$$
$$\{x_7, y_4, y_{11}, y_{12}, y_{13}\}, \quad \{x_8, y_2, y_4, y_5, y_9\}, \quad \{x_9, y_3, y_4, y_6, y_8\},$$
$$\{x_{10}, y_1, y_8, y_9, y_{13}\}, \quad \{x_{11}, y_1, y_2, y_6, y_{12}\}, \quad \{x_{12}, y_1, y_3, y_5, y_{11}\},$$
$$\{x_{13}, y_1, y_4, y_7, y_{10}\}.$$

Let B_3 be the collection of ten blocks in a covering of the pairs of $\{x_1, x_2, ..., x_{13}\}$ with blocks of size 5.

The collection of blocks $B_1 \cup B_2 \cup B_3$ forms a covering of pairs of W

1, 2, 3	5, 16, 21	9, 13, 20	19, 23, 27			10, 15, 17	6, 14, 22	8, 18, 25		4, 12, 26	7, 11, 24	
4, 5, 6	1, 15, 26		20, 24, 25	7, 13, 19	8, 11, 23	12, 14, 16	2, 10, 21			3, 17, 22		9, 18, 27
7, 8, 9		1, 17, 24	21, 22, 26	6, 12, 27	4, 16, 19	11, 13, 18		3, 10, 20			2, 15, 25	4, 14, 23
10, 11, 12			1, 5, 9	3, 18, 24	2, 14, 26		8, 16, 27	6, 13, 23	19, 22, 25	7, 15, 20	4, 17, 20	
13, 14, 15	7, 12, 23	6, 10, 26	3, 4, 8	1, 16, 22			9, 17, 25		21, 24, 27		5, 18, 19	2, 11, 20
16, 17, 18	8, 10, 24	4, 11, 27	2, 6, 7		1, 13, 25			5, 15, 22	20, 23, 26	9, 14, 19		3, 12, 21
19, 20, 21	3, 14, 25	2, 18, 22		5, 11, 26	9, 12, 24	1, 6, 8	4, 15, 23	7, 17, 27	10, 13, 16			
22, 23, 24		8, 15, 19			6, 18, 21	2, 4, 9	1, 12, 20		11, 14, 17	5, 10, 27	3, 13, 26	7, 16, 25
25, 26, 27	6, 17, 19			8, 14, 20		3, 5, 7		1, 11, 21	12, 15, 18	2, 16, 24	9, 10, 23	4, 13, 22
	2, 13, 27	3, 16, 23	10, 14, 18	9, 15, 21	5, 17, 20	19, 24, 26			1, 4, 7	6, 11, 25	8, 12, 22	
		7, 14, 21	11, 15, 16	4, 10, 25		20, 22, 27	5, 13, 24	2, 12, 19	3, 6, 9	1, 18, 23		8, 17, 26
	4, 18, 20		12, 13, 17		7, 10, 22	21, 23, 25	3, 11, 19	9, 16, 26	2, 5, 8		1, 14, 27	6, 15, 24
	9, 11, 22	5, 12, 25		2, 17, 23	3, 15, 27		7, 18, 26	4, 14, 24		8, 13, 21	6, 16, 20	1, 10, 19

FIG. 6.2. DR(27, 3, 1)-BIBD.

with 140 blocks of size 5. Since $L(53, 5, 2) = 138$, we have shown that $C(53, 5, 2) \leqslant L(53, 5, 2) + 2$. The lower bound $C(53, 5, 2) \geqslant L(53, 5, 2) + 1$ was proved in [2]. ∎

This construction is a special case of the next result which is proved in [8]. We will need the following definition. Let x be a positive integer and $V = \{1, 2, \ldots, 24x + 3\}$. Let D be the $(12x + 1) \times (12x + 1)$ array constructed from a DR$(24x + 3, 3, 1)$-BIBD defined on V. Add a new element y_i to each empty cell in column i in D for $i = 1, 2, \ldots, 12x + 1$. Row i of D now contains a subset S_i of cardinality $4x$ of $Y = \{y_1, y_2, \ldots, y_{12x+1}\}$. Partition each subset S_i into x subsets of size 4 for $i = 1, 2, \ldots, 12x + 1$. If this can be done so that the collection of 4 subsets formed is a $(12x + 1, 4, 1)$-BIBD defined on Y, then we say that the complement of D is a $(12x + 1, 4, 1)$-BIBD.

TABLE I

$C(v,5,2)$ for $v \equiv 1$ and 2 modulo 4

v	$C(v,5,2)$	Restrictions on v	Reference
$100n+1$	$L(v,5,2)$	$n \geq 1$	BIBD
$100n+2$	$L(v,5,2)$	$n \geq 1$	Theorem 1.1
$100n+5$	$L(v,5,2)$	$n \geq 0$	BIBD
$100n+6$	$L(v,5,2)$	$n \geq 0$	Theorem 1.1
$100n+9$	$L(v,5,2)$	$n \geq 24$	Lemma 5.4 (i)
$100n+10$	$L(v,5,2)$	$n \geq 0$	Lemma 1.2
$100n+13$	$L(v,5,2)+1$	$n \geq 1$	Lemma 1.2
$100n+14$	$L(v,5,2)$	$n \geq 0$	Lemma 1.2
$100n+17$	$L(v,5,2)$	$n \geq 1$	Gardner [2], Lemma 1.2
$100n+18$	$L(v,5,2)$	$n \geq 0$	Gardner [2], Lemma 1.2
$100n+21$	$L(v,5,2)$	$n \geq 0$	BIBD
$100n+22$	$L(v,5,2)$	$n \geq 0$	Theorem 1.1
$100n+25$	$L(v,5,2)$	$n \geq 0$	BIBD
$100n+26$	$L(v,5,2)$	$n \geq 0$	Theorem 1.1
$100n+29$	$L(v,5,2)$	$n \geq 134$	Lemma 5.4 (ii)
$100n+30$	$L(v,5,2)$	$n \geq 0$	Theorem 5.6
$100n+33$	$L(v,5,2)+1$	$n \geq 7$	Lemma 5.7 (i)
$100n+34$	$L(v,5,2)$	$n \geq 0$	Lemma 5.9 (i)
$100n+37$	$L(v,5,2)$	$n \geq 7$	Lemma 5.11 (i)
$100n+38$	$L(v,5,2)$	$n \geq 0$	Theorem 5.13
$100n+41$	$L(v,5,2)$	$n \geq 0$	BIBD
$100n+42$	$L(v,5,2)$	$n \geq 0$	Theorem 1.1
$100n+45$	$L(v,5,2)$	$n \geq 0$	BIBD
$100n+46$	$L(v,5,2)$	$n \geq 0$	Theorem 1.1
$100n+49$	$L(v,5,2)$	$n \geq 134$	Lemma 5.4 (iii)
$100n+50$	$L(v,5,2)$	$n \geq 0$	Theorem 5.6
$100n+53$	$L(v,5,2)+1$	$n \geq 27$	Lemma 5.7 (ii)
$100n+54$	$L(v,5,2)$	$n \geq 0$	Lemma 5.9 (ii)
$100n+57$	$L(v,5,2)$	$n \geq 33$	Lemma 5.11 (ii)
$100n+58$	$L(v,5,2)$	$n \geq 0$	Theorem 5.13
$100n+61$	$L(v,5,2)$	$n \geq 0$	BIBD
$100n+62$	$L(v,5,2)$	$n \geq 0$	Theorem 1.1
$100n+65$	$L(v,5,2)$	$n \geq 0$	BIBD
$100n+66$	$L(v,5,2)$	$n \geq 0$	Theorem 1.1
$100n+69$	$L(v,5,2)$	$n \geq 28$	Lemma 5.4 (iv)
$100n+70$	$L(v,5,2)$	$n \geq 0$ $n \neq 2$	Theorem 5.6
$100n+73$	$L(v,5,2)+1$	$n \geq 22$	Lemma 5.7 (iii)
$100n+74$	$L(v,5,2)$	$n \geq 0$ $n \neq 2$	Lemma 5.9 (iii)
$100n+77$	$L(v,5,2)$	$n \geq 33$	Lemma 5.11 (iii)
$100n+78$	$L(v,5,2)$	$n \geq 0$	Theorem 5.13
$100n+81$	$L(v,5,2)$	$n \geq 0$	BIBD
$100n+82$	$L(v,5,2)$	$n \geq 0$	Theorem 1.1
$100n+85$	$L(v,5,2)$	$n \geq 0$	BIBD
$100n+86$	$L(v,5,2)$	$n \geq 0$	Theorem 1.1
$100n+89$	$L(v,5,2)$	$n \geq 23$	Lemma 5.4 (v)
$100n+90$	$L(v,5,2)$	$n \geq 0$	Theorem 5.6
$100n+93$	$L(v,5,2)+1$	$n \geq 0$	Lemma 1.2
$100n+94$	$L(v,5,2)$	$n \geq 0$	Lemma 1.2
$100n+97$	$L(v,5,2)$	$n \geq 0$	Gardner [2], Lemma 1.2
$100n+98$	$L(v,5,2)$	$n \geq 0$	Gardner [2], Lemma 1.2

LEMMA 6.3 [8]. *Let b be a positive integer and let D be the array constructed from a* DR$(24n+3, 3, 1)$-BIBD. *If the complement of D is a* $(12n+1, 4, 1)$-BIBD *and if $C(12n+1, 5, 2) = L(12n+1, 5, 2) + s$ where s is a nonnegative integer, then $C(48n+5, 5, 2) \leqslant L(48n+5, 5, 2) + s$.*

7. SUMMARY

In this section, we summarize the known results for covering the pairs of a v element set with quintuples when $v \equiv 1$ or 2 modulo 4. For $v \equiv 2$ modulo 4 we have proved the following:

THEOREM 7.1. *If $v \equiv 2 \pmod 4$ and $v \neq 270, 274$, then $C(v, 5, 2) = L(v, 5, 2)$.*

In Table I, we list the values of v according to their congruence class modulo 100 for which $C(v, 5, 2)$ has been determined. In this listing, n denotes a nonnegative integer.

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